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AN EQUATION FOR THE PROBABILITY DENSITY, VELOCITY, AND TEMPERATURE OF PARTICLES IN A TURBULENT FLOW MODELLED BY A RANDOM GAUSSIAN FIELD*

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The motion and heat exchange of a dispersed admixture of solid particles suspended in turbulent flow are considered on the assumption that the volume concentration of the particles is small, so that the role of collisions between the particles is negligible. The fluctuating motion of the particles is governed by viscous drag from the surrounding turbulent flow and by forces of molecular origin which produce Brownian motion of the particles.

Fluctuations in the particle temperature are caused by fluctuations in the heat flux to the particles in the random temperature field of the fluid phase. The turbulent random velocity and temperature fields of the carrier phase are modelled by a Gaussian random process with a given autocorrelation function. In spite of the fact that describing a real turbulent flow by a Gaussian process is a somewhat approximate procedure, this approach, by virtue of its simplicity, is widely used to construct equations for probability density distributions for turbulent flow velocity, and also for studying the turbulent diffusion of passive admixtures, and hence for admixtures of inertial particles /1-8/. Brownian motion of the particles is modelled by a Gaussian process that is δ-correlated with time.

A closed equation for the probability density functions (PDFs) of the velocity and temperature of particles in inhomogeneous turbulent flow is constructed using the method of functional differentiation; on the basis of the PDF equations a system of equations for the first and second moments of the fluctuations of the dynamic and thermal characteristics of the solid phase is obtained.

1. The PDF equation. The equations of motion for a single solid particle have the form

$$dR_{pi}/dt = V_{pi}(t) \tag{1.1}$$

$$dV_{pi}/dt = \tau_u^{-1}(U_i(\mathbf{R}_p, t) - V_{pi}) + F_i(\mathbf{R}_p, t) + f_i(\mathbf{R}_p, t) \tag{1.2}$$

where R_{pi} and V_{pi} are the coordinates and velocity of the particle, $U_i(\mathbf{x}, t)$ is the velocity of the carrier flux, τ_u is the dynamic relaxation time for the particles in the Stokes approximation, $F_i(\mathbf{x}, t)$ is the bulk force acting on the particle, (for example, the force of gravity), and $f_i(\mathbf{x}, t)$ describes the random force imparting Brownian motion to the particle. Eq.(1.2) is written on the assumption that the density of the fluid phase is substantially smaller than the density of the material of the particle, so that forces governed by the pressure gradient in the fluid and the adjoining mass, and also the Bass force in the particle equations of motion, can be ignored. The heat-exchange equation for a single particle has the form

$$d\Theta_p/dt = \tau_\theta^{-1}(\Theta_1(\mathbf{R}_p, t) - \Theta_p) \tag{1.3}$$

where Θ_p is the particle temperature, $\Theta_1(\mathbf{x}, t)$ is the temperature of the carrier phase, and τ_θ is the thermal relaxation time.

Expressions (1.2) and (1.3) are Langevin equations in which the random field of Brownian forces is considered to be independent of the turbulent velocity fluctuations in the velocity and temperature of the carrier phase.

We will represent the velocity and temperature of the fluid phase in the form of the sum of averaged and fluctuating terms:

$$U_i(\mathbf{x}, t) = \langle U_i(\mathbf{x}, t)\rangle + u_i(\mathbf{x}, t),\ \langle u_i(\mathbf{x}, t)\rangle = 0$$

$$\Theta_1(\mathbf{x}, t) = \langle \Theta_1(\mathbf{x}, t)\rangle + \theta_1(\mathbf{x}, t),\ \langle \theta_1(\mathbf{x}, t)\rangle = 0$$

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We introduce a PDF for the particles with respect to coordinates, velocity and temperature:

$$\langle \Phi(\mathbf{x}, \mathbf{V}, \Theta, t)\rangle = \langle \delta(\mathbf{x} - \mathbf{R}_p)\delta(\mathbf{V} - \mathbf{V}_p)\delta(\Theta - \Theta_p)\rangle \tag{1.4}$$

The averaging is carried out over samples of the turbulent fluctuation field and the random force field **f**. Differentiating (1.4) with respect to time, using Eqs.(1.1)-(1.3) and the properties of the δ-function, we obtain

$$\frac{\partial\langle\Phi\rangle}{\partial t} + V_k\frac{\partial\langle\Phi\rangle}{\partial x_k} + \frac{1}{\tau_u}\frac{\partial}{\partial V_k}\left[(\langle U_k\rangle + \tau_u F_k - V_k)\langle\Phi\rangle\right] + \tag{1.5}$$

$$\frac{1}{\tau_\theta}\frac{\partial}{\partial\Theta}\left[(\langle\Theta_1\rangle - \Theta)\langle\Phi\rangle\right] = -\frac{\partial}{\partial V_k}\left[\frac{1}{\tau_u}\langle u_k\Phi\rangle + \langle f_k\Phi\rangle\right] - \frac{1}{\tau_\theta}\frac{\partial}{\partial\Theta}\langle\theta_1\Phi\rangle$$

To obtain a closed equation for $\langle\Phi\rangle$ it is necessary to find expressions for the correlations $\langle u_k\Phi\rangle$, $\langle f_k\Phi\rangle$, and $\langle\theta_1\Phi\rangle$. To this end we consider the fields **u**, **f** and θ to be Gaussian, using the Furutsu-Novikov formula /9/

$$\langle z(\mathbf{x})R[z]\rangle = \int d\mathbf{x}_1\langle z(\mathbf{x})z(\mathbf{x}_1)\rangle\left\langle\frac{\delta R[z(\mathbf{x})]}{\delta z(\mathbf{x}_1)}\right\rangle \tag{1.6}$$

where $z(\mathbf{x})$ is a stochastic process in **x** space, $R[z]$ is a functional of the stochastic process z and $\delta R/\delta z$ is a functional derivative.

By (1.6) and (1.4) we have

$$\langle u_k\Phi\rangle = \int d\mathbf{x}_1\int_0^\infty d\xi\langle u_i(\mathbf{x}_1,\xi)u_k(\mathbf{x},t)\rangle\left\langle\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta u_i(\mathbf{x}_1,\xi)}\right\rangle + \tag{1.7}$$

$$\int d\mathbf{x}_1\int_0^\infty d\xi\langle\theta_1(\mathbf{x}_1,\xi)u_k(\mathbf{x},t)\rangle\left\langle\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta\theta_1(\mathbf{x}_1,\xi)}\right\rangle$$

$$\langle\theta_1\Phi\rangle = \int d\mathbf{x}_1\int_0^\infty d\xi\langle\theta_1(\mathbf{x}_1,\xi)\theta_1(\mathbf{x},t)\rangle\left\langle\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta\theta_1(\mathbf{x}_1,\xi)}\right\rangle +$$

$$\int d\mathbf{x}_1\int_0^\infty d\xi\langle u_j(\mathbf{x}_1,\xi)\theta_1(\mathbf{x},t)\rangle\left\langle\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta u_i(\mathbf{x}_1,\xi)}\right\rangle$$

$$\langle f_k\Phi\rangle = \int d\mathbf{x}_1\int_0^\infty d\xi\langle f_i(\mathbf{x}_1,\xi)f_k(\mathbf{x},t)\rangle\left\langle\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta f_i(\mathbf{x}_1,\xi)}\right\rangle \tag{1.8}$$

where, for example, the functional derivative of Φ with respect to the function $u_i(\mathbf{x}_1,\xi)$ has the form

$$\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta u_i(\mathbf{x}_1,\xi)} = -\frac{\partial}{\partial x_j}\Phi\frac{\delta R_{pj}(t)}{\delta u_i(\mathbf{x}_1,\xi)} - \frac{\partial}{\partial V_j}\Phi\frac{\delta V_{pj}(t)}{\delta u_i(\mathbf{x}_1,\xi)} - \frac{\partial}{\partial\Theta}\Phi\frac{\delta\Theta_p(t)}{\delta u_i(\mathbf{x}_1,\xi)} \tag{1.9}$$

The functional derivatives of Φ with respect to other random functions can be expressed similarly. To compute functional derivatives with respect to particle coordinates, velocity and temperature, we will write the equations of motion and heat exchange of a single particle in the integral form

$$R_{pi}(t) = R_{pi}(0) + \int_0^t d\xi V_{pi}(\xi) \tag{1.10}$$

$$V_{pi}(t) = V_{pi}(0)\exp\left(-\frac{t}{\tau_u}\right) + \frac{1}{\tau_u}\int_0^t d\xi\exp\left(-\frac{t-\xi}{\tau_u}\right)\times$$

$$[U_i(\mathbf{R}_p(\xi),\xi) + \tau_u F_i(\mathbf{R}_p(\xi),\xi) + \tau_u f_i(\mathbf{R}_p(\xi),\xi)]$$

$$\Theta_p(t) = \Theta_p(0)\exp\left(-\frac{t}{\tau_\theta}\right) + \frac{1}{\tau_\theta}\int_0^t d\xi\exp\left(-\frac{t-\xi}{\tau_\theta}\right)\Theta_1(\mathbf{R}_p(\xi),\xi)$$

We apply the functional differentiation operator $\delta/\delta u_i(\mathbf{x}_1,\xi)$ to (1.10). Taking into account the fact that $\delta u_j(\mathbf{x},t)/\delta u_i(\mathbf{x}_1,\xi) = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}_1)\delta(t-\xi)$, the causality principle /9/ and the independence of the initial values of the particle coordinates, velocity and temperature from the random fields, we obtain a system of integral equations for finding the functional derivatives:

$$\frac{\delta V_{pi}(t)}{\delta u_j(\mathbf{x}_1,\xi)} = \frac{\delta_{ij}}{\tau_u}\exp\left(-\frac{t-\xi}{\tau_u}\right)\delta(\mathbf{x}_1-\mathbf{R}_p(\xi)) + \tag{1.11}$$
$$\frac{1}{\tau_u}\int_\xi^t ds\exp\left(-\frac{t-s}{\tau_u}\right)\left[\frac{\partial U_i(\mathbf{R}_p(s),s)}{\partial x_m} + \tau_u\frac{\partial F_i(\mathbf{R}_p(s),s)}{\partial x_m} + \tau_u\frac{\partial f_i(\mathbf{R}_p(s),s)}{\partial x_m}\right]\frac{\delta R_{pm}(s)}{\delta u_j(\mathbf{x}_1,\xi)}$$
$$\frac{\delta R_{pi}(t)}{\delta u_j(x_1,\xi)} = \delta_{ij}\left[1-\exp\left(-\frac{t-\xi}{\tau_u}\right)\right]\delta(\mathbf{x}_1-\mathbf{R}_p(\xi)) +$$
$$\int_\xi^t ds\left[1-\exp\left(-\frac{t-s}{\tau_u}\right)\right]\left[\frac{\partial U_i(\mathbf{R}_p(s),s)}{\partial x_m} + \tau_u\frac{\partial F_i(\mathbf{R}_p(s),s)}{\partial x_m} + \tau_u\frac{\partial f_i(\mathbf{R}_p(s),s)}{\partial x_m}\right]\frac{\delta R_{pm}(s)}{\delta u_j(\mathbf{x}_1,\xi)}$$
$$\frac{\delta\Theta_p(t)}{\delta u_j(\mathbf{x}_1,\xi)} = \frac{1}{\tau_\theta}\int_\xi^t ds\exp\left(-\frac{t-s}{\tau_\theta}\right)\frac{\partial\theta_1(\mathbf{R}_p(s),s)}{\partial x_m}\frac{\delta R_{pm}(s)}{\delta u_j(\mathbf{x}_1,\xi)}$$

It follows from (1.7) that a closed expression for $\langle u_k\Phi\rangle$ can only be obtained for a random field $u(\mathbf{x},t)$ that is δ-correlated in time:

$$\langle u_i(\mathbf{x}_1,t_1)u_j(\mathbf{x}_2,t_2)\rangle \sim \delta(t_1-t_2)$$

because in this case there are funtional derivatives at $\xi = t$ in the expression for $\langle u_k\Phi\rangle$. In the general case the autocorrelation function for turbulent fluctuations has a finite decay time. Here it is necessary to use an approximate solution for system (1.11). We introduce the parameter $\varepsilon = l/L$, where $l = T_E u$ (T_E is the characteristic decay time for the time correlation of the carrier flow and u is the characteristic value of the velocity fluctuations of the carrier flow), and L is the length scale for variations in the averaged quantities. In the case when $\varepsilon < 1$ the inhomogeneity of the turbulent flow, to within terms of the order of ε^2 can be taken into account using the first-order approximation to the system (1.11) /10, 11/

$$\frac{\delta V_{pi}(t)}{\delta u_j(\mathbf{x}_1,\xi)} = \frac{\delta_{ij}}{\tau_u}\exp\left(-\frac{t-\xi}{\tau_u}\right)\delta(\mathbf{x}_1-\mathbf{R}_p(\xi))$$
$$\frac{\delta R_{pi}(t)}{\delta u_j(\mathbf{x}_1,\xi)} = \delta_{ij}\left[1-\exp\left(-\frac{t-\xi}{\tau_u}\right)\right]\delta(\mathbf{x}_1-\mathbf{R}_p(\xi)),\quad \frac{\delta\Theta_p(t)}{\delta u_j(\mathbf{x}_1,\xi)} = 0$$

Similarly, to within terms of order ε^2 one can compute the functional derivatives of the particle velocity, coordinates and temperature with respect to the temperature fluctuations of the carrier phase. As a result we obtain expressions for the correlations $\langle u_k\Phi\rangle$ and $\langle\theta_1\Phi\rangle$

$$\langle u_k\Phi\rangle = -\tau_u g_{uu}\langle u_i u_k\rangle\frac{\partial\langle\Phi\rangle}{\partial x_i} - q_{uu}\langle u_i u_k\rangle\frac{\partial\langle\Phi\rangle}{\partial V_i} - q_{\theta u}\langle\theta_1 u_i\rangle\frac{\partial\langle\Phi\rangle}{\partial\Theta} \tag{1.12}$$
$$\langle\theta_1\Phi\rangle = -q_{\theta\theta}\langle\theta_1^2\rangle\frac{\partial\langle\Phi\rangle}{\partial\Theta} - \tau_u g_{u\theta}\langle\theta_1 u_i\rangle\frac{\partial\langle\Phi\rangle}{\partial x_i} - q_{u\theta}\langle\theta_1 u_i\rangle\frac{\partial\langle\Phi\rangle}{\partial V_i}$$

The functional derivative of Φ with respect to the temporally δ-correlated random field f is equal to

$$\langle f_i\Phi\rangle = \int_0^\infty d\xi\int d\mathbf{x}_1\langle f_1(\mathbf{x},t)f_j(\mathbf{x}_1,\xi)\rangle\left\langle\frac{\delta\Phi(\mathbf{x},\mathbf{V},\Theta,t)}{\delta f_j(\mathbf{x}_1,\xi)}\right\rangle = \frac{D\delta_{ij}}{\tau_u^2}\frac{\partial\langle\Phi\rangle}{\partial V_j} \tag{1.13}$$

where D is the Brownian diffusion coefficient of the particles.

The coefficients q and g are found from the relations

$$\frac{1}{\tau_u}\int_0^t d\xi\exp\left(-\frac{t-\xi}{\tau_u}\right)\langle u_i(\mathbf{R}_p(\xi),\xi)u_k(\mathbf{x},t)\rangle = q_{uu}(\mathbf{x},t)\langle u_i(\mathbf{x},t)u_k(\mathbf{x},t)\rangle \tag{1.14}$$
$$\frac{1}{\tau_u}\int_0^t d\xi\left[1-\exp\left(-\frac{t-\xi}{\tau_u}\right)\right]\langle u_i(\mathbf{R}_p(\xi),\xi)u_k(\mathbf{x},t)\rangle = g_{uu}(\mathbf{x},t)\langle u_i(\mathbf{x},t)u_k(\mathbf{x},t)\rangle$$
$$\frac{1}{\tau_\theta}\int_0^t d\xi\exp\left(-\frac{t-\xi}{\tau_\theta}\right)\langle\theta_1(\mathbf{R}_p(\xi),\xi)u_k(\mathbf{x},t)\rangle = q_{\theta u}(\mathbf{x},t)\langle\theta_1(\mathbf{x},t)u_k(\mathbf{x},t)\rangle$$

$$\frac{1}{\tau_\theta}\int_0^t d\xi \exp\left(-\frac{t-\xi}{\tau_\theta}\right)\langle\theta_1(\mathbf{R}_p(\xi),\xi)\,\theta_1(\mathbf{x},t)\rangle = g_{\theta\theta}(\mathbf{x},t)\langle\theta_1(\mathbf{x},t)\,\theta_1(\mathbf{x},t)\rangle$$

$$\frac{1}{\tau_u}\int_0^t d\xi (\exp\left(-\frac{t-\xi}{\tau_u}\right)\langle u_i(\mathbf{R}_p(\xi),\xi)\,\theta_1(\mathbf{x},t)\rangle = q_{u\theta}(\mathbf{x},t)\langle\theta_1(\mathbf{x},t)\,u_i(\mathbf{x},t)\rangle$$

As can be seen from formulae (1.14), the entrainment coefficients g and q of the particles in the turbulent fluctuations of the carrier phase and determined by the Lagrangian fluctuation correlations of the characteristics of the carrier phase computed along the particle trajectories. The two-point correlation moments which occur in expressions (1.14), as in /12, 13/, are approximated by the step function

$$\frac{\langle u_i(\mathbf{R}_p(\xi),\xi)\,u_k(\mathbf{x},t)\rangle}{\langle u_i u_k\rangle} = \frac{\langle\theta_1(\mathbf{R}_p(\xi),\xi)\,u_i(\mathbf{x},t)\rangle}{\langle\theta_1 u_i\rangle} = \frac{\langle u_i(\mathbf{R}_p(\xi),\xi)\,\theta_1(\mathbf{x},(t)\rangle}{\langle\theta_1 u_i\rangle} = \tag{1.15}$$
$$\frac{\langle\theta_1(\mathbf{R}_p(\xi),\xi)\,\theta_1(\mathbf{x},t)\rangle}{\langle\theta_1^2\rangle} = \begin{cases}1, & |t-\xi|\leqslant T_p\\ 0, & |t-\xi|>T_p\end{cases}$$

Here T_p is a characteristic interaction time for the particles with the turbulent field, given in the form

$$T_p = \int_0^\infty d\xi\,\frac{\langle u_i(\mathbf{R}_p(\xi),\xi)\,u_j(\mathbf{x},t)\rangle}{\langle u_i u_j\rangle}$$

To a first approximation the time T_p can be taken to be equal to the integral time-scale for the turbulence. Using (1.15) we obtain from (1.14) the following expressions for the entrainment coefficients of particles in the fluctuating motion

$$q_{uu} = q_{u\theta} = 1 - \exp(-1/\Omega_u),\quad q_{\theta\theta} = q_{\theta u} = 1 - \exp(-1/\Omega_\theta)$$
$$g_{uu} = g_{u\theta} = 1/\Omega_u - 1 + \exp(-1/\Omega_u),\quad \Omega_u = \tau_u/T_p,\quad \Omega_\theta = \tau_\theta/T_p$$

Substituting expressions (1.12)-(1.13) into (1.5), we obtain a closed equation for the PDF of the particles in the turbulent flow:

$$\frac{\partial\langle\Phi\rangle}{\partial t} + V_k\frac{\partial\langle\Phi\rangle}{\partial x_k} + \frac{1}{\tau_u}\frac{\partial}{\partial V_k}[(\langle U_k\rangle + F_k - V_k)\langle\Phi\rangle] + \tag{1.16}$$
$$\frac{1}{\tau_\theta}\frac{\partial}{\partial\theta}[(\langle\Theta_1\rangle - \Theta)\langle\Phi\rangle] = g_{uu}\langle u_i u_k\rangle\frac{\partial^2\langle\Phi\rangle}{\partial x_i\partial V_k} +$$
$$\frac{1}{\tau_u}\left(q_{uu}\langle u_i u_k\rangle + \frac{D\delta_{ik}}{\tau_u}\right)\frac{\partial^2\langle\Phi\rangle}{\partial V_i\partial V_k} + \frac{\tau_u}{\tau_\theta}g_{u\theta}\langle\theta_1 u_i\rangle\frac{\partial^2\langle\Phi\rangle}{\partial\Theta\partial x_i} +$$
$$\left(\frac{q_{u\theta}}{\tau_\theta} + \frac{q_{\theta u}}{\tau_u}\right)\langle\theta_1 u_i\rangle\frac{\partial^2\langle\Phi\rangle}{\partial\Theta\partial V_i} + \frac{1}{\tau_\theta}q_{\theta\theta}\langle\theta_1^2\rangle\frac{\partial^2\langle\Phi\rangle}{\partial\Theta^2}$$

Integrating Eq.(1.16) with respect to temperature, we can write down the equation for the particle PDF with respect to coordinates and velocity $\langle\Phi_V(\mathbf{x},\mathbf{V},t)\rangle = \int d\Theta\,\langle\Phi(\mathbf{x},\mathbf{V},\Theta,t)\rangle$

$$\frac{\partial\langle\Phi_V\rangle}{\partial t} + V_k\frac{\partial\langle\Phi_V\rangle}{\partial x_k} + \frac{1}{\tau_u}\frac{\partial}{\partial V_k}[(\langle U_k\rangle + F_k - V_k)\langle\Phi_V\rangle] = \tag{1.17}$$
$$g_{uu}\langle u_i u_k\rangle\frac{\partial^2\langle\Phi_V\rangle}{\partial x_i\partial V_k} + \frac{1}{\tau_u}\left(q_{uu}\langle u_i u_k\rangle + \frac{D\delta_{ik}}{\tau_u}\right)\frac{\partial^2\langle\Phi_V\rangle}{\partial V_i\partial V_k}$$

For laminar flow $(\langle u_i u_k\rangle = 0)$ Eq.(1.17) reduces to the Fokker-Planck equation for Brownian particles /14/. For highly inertial particles $\Omega_u \gg 1$ $(g_{uu} \sim 1/\Omega_u^2,\ q_{uu} \sim 1/\Omega_u)$ Eq.(1.17) agrees with the PDF equations obtained in /1/. We note that the PDF equations in /1/ can be obtained assuming temporal δ-correlation of the carrier flow's turbulent velocity field fluctuations.

2. ***Equations for the moments.*** We define the averaged number denisty, velocity and temperature of the discrete phase

$$\langle N(\mathbf{x},\ )\rangle = \int d\mathbf{V}d\Theta\,\langle\Phi(\mathbf{x},\mathbf{V},\Theta,t)\rangle$$
$$\langle N(\mathbf{x},t)\rangle\langle V_i(\mathbf{x},t)\rangle = \int d\mathbf{V}d\Theta V_i\,\langle\Phi(\mathbf{x},\mathbf{V},\Theta,t)\rangle$$
$$\langle N(\mathbf{x},t)\rangle\langle\Theta_2(\mathbf{x},t)\rangle = \int d\mathbf{V}d\Theta\Theta\,\langle\Phi(\mathbf{x},\mathbf{V},\Theta,t)\rangle$$

In this system of notation for the moments, the PDF Eq.(1.16) can be conveniently written using variables $\mathbf{x}, \mathbf{v}, \theta_2$ and t, (where $\mathbf{v} = \mathbf{V} - \langle \mathbf{V} \rangle$ and $\theta = \Theta - \langle \Theta_2 \rangle$ are the velocity and temperature fluctuations of the discrete phase):

$$\begin{gathered}
\frac{\partial \langle \Phi \rangle}{\partial t} + \langle V_k \rangle \frac{\partial \langle \Phi \rangle}{\partial x_k} - \left(\frac{\partial \langle V_i \rangle}{\partial t} + \langle V_k \rangle \frac{\partial \langle V_i \rangle}{\partial x_k} \right) \frac{\partial \langle \Phi \rangle}{\partial v_i} - \\
\left(\frac{\partial \langle \Theta_2 \rangle}{\partial t} + \langle V_k \rangle \frac{\partial \langle \Theta_2 \rangle}{\partial x_k} \right) \frac{\partial \langle \Phi \rangle}{\partial \theta} + \frac{1}{\tau_u} \frac{\partial}{\partial v_i} [(\langle U_i \rangle + F_i - \langle V_i \rangle) \langle \Phi \rangle] + \\
\frac{1}{\tau_\theta} \frac{\partial}{\partial \theta} [(\langle \Theta_1 \rangle - \langle \Theta_2 \rangle) \langle \Phi \rangle] - \frac{1}{\tau_u} \frac{\partial}{\partial v_i} v_i \langle \Phi \rangle - \frac{1}{\tau_\theta} \frac{\partial}{\partial \theta} \theta \langle \Phi \rangle + \\
v_k \frac{\partial \langle \Phi \rangle}{\partial x_k} - v_k \frac{\partial \langle V_i \rangle}{\partial x_k} \frac{\partial \langle \Phi \rangle}{\partial v_i} - v_k \frac{\partial \langle \Theta_2 \rangle}{\partial x_k} \frac{\partial \langle \Phi \rangle}{\partial \theta} = \\
\frac{1}{\tau_n} \left(q_{uu} \langle u_j u_k \rangle + \frac{D \delta_{jk}}{\tau_u} - \tau_u g_{uu} \langle u_i u_k \rangle \frac{\partial \langle V_j \rangle}{\partial x_k} \right) \frac{\partial^2 \langle \Phi \rangle}{\partial v_j \partial v_k} + \\
\frac{1}{\tau_\theta} \left(q_{\theta\theta} \langle \theta_1^2 \rangle - \tau_u g_{u\theta} \langle \theta_1 u_i \rangle \frac{\partial \langle \Theta_2 \rangle}{\partial x_i} \frac{\partial^2 \langle \Phi \rangle}{\partial \theta^2} + g_{uu} \langle u_j u_k \rangle \frac{\partial^2 \langle \Phi \rangle}{\partial v_j \partial x_k} + \right. \\
\frac{\tau_u}{\tau_\theta} g_{u\theta} \langle \theta_1 u_i \rangle \frac{\partial^2 \langle \Phi \rangle}{\partial \theta \partial x_i} + \left[\frac{1}{\tau_\theta} \left(g_{u\theta} \langle \theta_1 u_j \rangle - \tau_u g_{u\theta} \langle \theta_1 u_i \rangle \frac{\partial \langle V_j \rangle}{\partial x_i} \right) + \right. \\
\left. \frac{1}{\tau_u} \left(g_{\theta u} \langle \theta_1 u_j \rangle - \tau_u g_{uu} \langle u_i u_j \rangle \frac{\partial \langle \Theta_2 \rangle}{\partial x_i} \right) \right] \frac{\partial^2 \langle \Phi \rangle}{\partial v_j \partial \theta}
\end{gathered} \tag{2.1}$$

Integrating Eq.(2.1) over velocity space and temperature, we obtain an equation for the averaged particle number density

$$\frac{\partial \langle N \rangle}{\partial t} + \frac{\partial}{\partial x_k} \langle N \rangle \langle V_k \rangle = 0 \tag{2.2}$$

Multiplying (2.1) by v_i and integrating over velocity space and temperature, we obtain an equation for the average velocity of the solid phase

$$\frac{\partial \langle V_i \rangle}{\partial t} + \langle V_k \rangle \frac{\partial \langle V_i \rangle}{\partial x_k} + \frac{\partial \langle v_i v_k \rangle}{\partial x_k} = \frac{1}{\tau_u} (\langle U_i \rangle + F_i - \langle V_i \rangle) - \frac{D_{ik}}{\tau_u} \frac{\partial \ln \langle N \rangle}{\partial x_k} \tag{2.3}$$

$$D_{ik} = \tau_u (\langle v_i v_k \rangle + g_{uu} \langle u_i u_k \rangle)$$

where $\langle v_i v_k \rangle \langle N \rangle = \int dv d\theta v_i v_k \langle \Phi \rangle$ is the Reynolds stress tensor appearing in the solid phase as a result of the particles participating in the fluctuating motion, and D_{ik} is the turbulent diffusion coefficient of the particles.

From Eqs.(2.2) and (2.3) we have a density balance equation (of hyperbolic type)

$$\begin{gathered}
\tau_u \frac{\partial^2 \langle N \rangle}{\partial t^2} + \frac{\partial \langle N \rangle}{\partial t} + \\
\frac{\partial}{\partial x_i} \left\{ \left[\langle U_i \rangle + F_i + \tau_u \frac{\partial}{\partial x_k} (\langle v_i v_k \rangle + \langle V_i \rangle \langle V_k \rangle) \right] \langle N \rangle \right\} = \\
\frac{\partial}{\partial x_i} \left[(D_{ik} + \tau_u \langle V_i \rangle \langle V_k \rangle) \frac{\partial \langle N \rangle}{\partial x_k} \right]
\end{gathered} \tag{2.4}$$

We note that the scattering process for the passive admixture is described by diffusion equations of parabolic type, for which the concentration of admixture at any instant of time even at an infinite distance from the source of the admixture (for diffusion in an unbounded medium) is non-zero. In the case of the scattering of an admixture of inertial particles the concentration of particles is localized in space, as a consquence of the hyperolic nature of Eq.(2.4).

Multiplying Eq.(2.1) by θ and integrating over velocity space and temperature, we obtain an equation for the average temperature of the solid phase

$$\frac{\partial \langle \Theta_2 \rangle}{\partial t} + \langle V_k \rangle \frac{\partial \langle \Theta_2 \rangle}{\partial x_k} + \frac{\partial \langle \theta_2 v_k \rangle}{\partial x_k} = \frac{1}{\tau_\theta} (\langle \Theta_1 \rangle - \langle \Theta_2 \rangle) - \frac{D_k^\theta}{\tau_\theta} \frac{\partial \ln \langle N \rangle}{\partial x_k}$$

$$D_k^\theta = \tau_\theta \langle \theta_2 v_k \rangle + \tau_u g_{u\theta} \langle \theta_1 u_k \rangle$$

where $\langle \theta_2 v_k \rangle \langle N \rangle = \int dv d\theta \theta v_k \langle \Phi \rangle$ is the turbulent heat flux appearing in the solid phase as a result of particle entrainment in the fluctuating motion, and D_k^θ is the coefficient of turbulent thermal diffusion.

Multiplying (2.1) by $v_i v_j$, θv_i, and θ^2 and integrating over velocity space and temperature, we can write down equations for the second moments of the velocity and temperature fluctuations of the discrete phase:

$$\frac{\partial\langle v_i v_j\rangle}{\partial t}+\langle V_k\rangle\frac{\partial\langle v_i v_j\rangle}{\partial x_k}+\frac{1}{\langle N\rangle}\frac{\partial\langle v_i v_j v_k\rangle\langle N\rangle}{\partial x_k}+ \tag{2.5}$$

$$\langle v_i v_k\rangle\frac{\partial\langle V_j\rangle}{\partial x_k}+\langle v_j v_k\rangle\frac{\partial\langle V_i\rangle}{\partial x_k}=\frac{1}{\tau_u}\left[2q_{uu}\langle u_i u_j\rangle+\frac{2D\delta_{ij}}{\tau_u}-\right.$$

$$\left.\tau_u g_{uu}\left(\langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_k}+\langle u_j u_k\rangle\frac{\partial\langle U_i\rangle}{\partial x_k}-2\langle v_i v_j\rangle\right]$$

$$\frac{\partial\langle\theta_2 v_i\rangle}{\partial t}+\langle V_k\rangle\frac{\partial\langle\theta_2 v_i\rangle}{\partial x_k}+\frac{1}{\langle N\rangle}\frac{\partial\langle\theta_2 v_i v_k\rangle\langle N\rangle}{\partial x_k}+\langle v_k\theta_2\rangle\frac{\partial\langle V_i\rangle}{\partial x_k}+$$

$$\langle v_i v_k\rangle\frac{\partial\langle\theta_2\rangle}{\partial x_k}=\frac{1}{\tau_\theta}\left[q_{v\theta}\langle\theta_1 u_i\rangle-\tau_u g_{u\theta}\langle\theta_1 u_k\rangle\frac{\partial\langle V_i\rangle}{\partial x_k}-\langle\theta_2 v_i\rangle\right]+ \tag{2.6}$$

$$\frac{1}{\tau_u}\left[q_{\theta u}\langle\theta_1 u_i\rangle-\tau_u g_{uu}\langle u_i u_k\rangle\frac{\partial\langle\theta_2\rangle}{\partial x_k}-\langle\theta_2 v_i\rangle\right]$$

$$\frac{\partial\langle\theta_2^2\rangle}{\partial t}+\langle V_k\rangle\frac{\partial\langle\theta_2^2\rangle}{\partial x_k}+\frac{1}{\langle N\rangle}\frac{\partial\langle\theta_2^2 v_k\rangle\langle N\rangle}{\partial x_k}+2\langle\theta_2 v_k\rangle\frac{\partial\langle\theta_2\rangle}{\partial x_k}= \tag{2.7}$$

$$\frac{2}{\tau_\theta}\left[q_{\theta\theta}\langle\theta_1^2\rangle-\tau_u g_{u\theta}\langle\theta_1 u_k\rangle\frac{\partial\langle\theta_1\rangle}{\partial x_k}-\langle\theta_2^2\rangle\right]$$

$$\langle v_i v_j v_k\rangle\langle N\rangle=\int d\theta d\mathbf{v}\, v_i v_j v_k\langle\Phi\rangle$$

$$\langle\theta_2 v_i v_k\rangle\langle N\rangle=\int d\mathbf{v} d\theta\,\theta_2 v_i v_k\langle\Phi\rangle$$

$$\langle\theta_2^2 v_k\rangle\langle N\rangle=\int d\mathbf{v} d\theta\,\theta_2^2 v_k\langle\Phi\rangle$$

where $\langle v_i v_j v_k\rangle$, $\langle\theta_2 v_i v_k\rangle$, and $\langle\theta_2^2 v_k\rangle$ are third moments of the velocity and temperature fluctuations of the discrete phase.

In the case of statistically stationary turbulence, and assuming small gradients in the averaged quantities, (2.5)-(2.7) imply the following relations between the second moments of the velocity and temperature fluctuations of the solid and carrier phases:

$$\langle v_i v_j\rangle=q_{uu}\langle u_i u_j\rangle+\frac{D}{\tau_u}\delta_{ij} \tag{2.8}$$

$$\langle\theta_2 v_i\rangle=\frac{\tau_u q_{u\theta}+\tau_\theta q_{\theta u}}{\tau_u+\tau_\theta}\langle\theta_1 u_i\rangle$$

$$\langle\theta_2^2\rangle=q_{\theta\theta}\langle\theta_1^2\rangle$$

From expressions (2.8) it is clear that small particles $\Omega_u, \Omega_\theta \ll 1$ are completely entrained in the fluctuating motion of the carrier phase $(q\to 1)$, whereas large (inertial) particles do not participate in the fluctuating motion $(q\to 0)$.

3. ***The approximation of stationary, homogeneous and isotropic turbulence.*** In this case $\langle u_i u_j\rangle = {}^2/_3 e\delta_{ij}$, $\langle\theta_1 u_i\rangle = 0$ (the velocity and temperature fields are not correlated with one another). It follows from (2.1) that the PDF for isotropic turbulence becomes a product of the normal velocity and temperature distributions

$$\langle\Phi\rangle=\langle N\rangle\left(\frac{3}{4\pi q_{uu}e}\right)^{3/2}\exp\left(-\frac{3v^2}{4q_{uu}e}\right)\frac{1}{(2\pi q_{\theta\theta}\langle\theta_1^2\rangle)^{1/2}}\exp\left(-\frac{\theta_2^2}{2q_{\theta\theta}\langle\theta_1^2\rangle}\right) \tag{3.1}$$

For small particles $(q\to 1)$ expression (3.1) becomes the PDF for the velocity and temperature of the turbulent carrier flow. The normal distribution for the PDF of velocity fluctuations of single-phase turbulent flow is established, for example, in /15/, while in /4/ it is shown that the normal law satisfactorily describes the distribution of admixture concentration and temperature in non-intermittent domains. As the particle inertia increases, distribution (3.1) tends to a δ-function:

$$\lim_{\substack{\Omega_u\to\infty\\ \Omega_\theta\to\infty}}\langle\Phi(\mathbf{x},\mathbf{V},\Theta,t)\rangle=\langle N\rangle\,\delta(\mathbf{V}-\langle\mathbf{V}(\mathbf{x},t)\rangle)\,\delta(\Theta-\langle\Theta_2(\mathbf{x},t)\rangle)$$

REFERENCES

1. KROSHILIN A.E., KUKHARENKO V.N. and NIGMATULIN B.I., Deposition of particles on a channel wall in turbulent dispersed gradient flow, Izv. Akad. Nauk SSSR, Mekhan. Zhidkosti i Gaza, 4, 1985.
2. REEKS M.W., Eulerian direct interaction applied to the statistical motion of particles in a turbulent fluid, J. Fluid Mech., 97, 3, 1980.
3. YUU S., Collision rate of small particles in a homogeneous and isotropic turbulence, Amer. Inst. Chem. Engng., 30, 5, 1984.
4. KUZNETSOV V.R. and SABEL'NIKOV V.A., Turbulence and Combustion, Nauka, Moscow, 1986.
5. POPE S.B., PDF methods for turbulent reactive flows, Progr. Energy Combust. Sci., 11, 2, 1985.
6. LUNGREN T.S. and POINTIN Y.B., Turbulent selfdiffusion, Phys. Fluids, 19, 3, 1976.
7. PHYTHIAN R., Dispersion by random velocity fields, J. Fluid Mech., 67, 1, 1975.
8. PISMEN L.M. and NIR A., On the motion of suspended particles in stationary homogeneous turbulence, J. Fluid Mech., 84, 1, 1978.
9. KLYATSKIN V.I., Stochastic Equations and Waves in Stochastically Inhomogeneous Media, Nauka, Moscow, 1980.
10. KLYATSKIN V.I. and TATARSKII V.I., On ray diffusion in a medium with random inhomogeneities, Izv. Vuz. Radiofizika, 14, 5, 1971.
11. MUZYCHUK O.V., A statistical description of linear systems with non-delta-correlated parameter fluctuations, Izv. Vuz. Radiofizika, 22, 10, 1979.
12. DEREVICH I.V., EROSHENKO V.M. and ZAICHIK L.I., The influence of particles on the intensity of turbulent transfer in a dust-carrying gas, Inzh.-Fiz. Zh., 45, 4, 1983.
13. DEREVICH I.V. and ZAICHIK L.I., The influence of particles on the intensity of the turbulent heat transfer, Inzh.-Fiz. Zh., 48, 4, 1985.
14. GARDINER K.V., Stochastic Methods in the Natural Sciences, Mir, Moscow, 1986.
15. CHENG P.M., A simplified statistical model for turbulence in a chemically-reacting flow with transverse shear, Raketnaya Tekhn. i Kosmonavtika, 7, 10, 1969.

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TRIPLE-WAVE POTENTIAL FLOWS OF A POLYTROPIC GAS*

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A system of equations describing triple-wave solutions for unsteady isentropic potential flows of a polytropic gas was derived in /1/. A family of exact triple-wave solutions of the equations of gas dynamics with three arbitrary functions of one argument was constructed in /2/ for $1<\gamma<2$. Some applications and properties of this family were studied. In this paper we show that the triple-wave equations of /1/ are a system in involution and depend on one arbitrary function of two arguments.

1. The equations of motion of polytropic gas in the unsteady isentropic case can be written in the form

$$du/dt + \nabla\theta = 0,\ d\theta/dt + \varkappa\theta \operatorname{div} \mathbf{u} = 0 \tag{1.1}$$
$$\theta = c^2/\varkappa,\ \varkappa = \gamma - 1 > 0,\ d/dt = \partial/\partial t + u_\alpha \partial/\partial x_\alpha$$

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